# Kink dynamics in a one-dimensional growing surface

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A high-symmetry crystal surface may undergo a kinetic instability during the growth, such that its late stage evolution resembles a phase separation process. This parallel is rigorous in one dimension, if the conserved surface current is derivable from a free energy. We study the problem in the presence of a physically relevant term breaking the up-down symmetry of the surface and that cannot be derived from a free energy. Following the treatment introduced by Kawasaki and Ohta [Physica A **116**, 573 (1982)] for the symmetric case, we are able to translate the problem of the surface evolution into a problem of nonlinear dynamics of kinks (domain walls). Because of the break of symmetry, two different classes ($A$ and $B$) of kinks appear and their analytical form is derived. The effect of the adding term is to shrink a kink $A$ and to widen the neighboring kink $B$ in such a way that the product of their widths keeps constant. Concerning the dynamics, this implies that kinks $A$ move much faster than kinks $B$. Since the kink profiles approach exponentially the asymptotical values, the time dependence of the average distance $L(t)$ between kinks does *not* change: $L(t)\sim \ln t$ in the absence of noise, and $L(t)\sim t^{1/3}$ in the presence of (shot) noise. However, the crossover time between the first and the second regime may increase even of some orders of magnitude. Finally, our results show that kinks $A$ may be so narrow that their width is comparable to the lattice constant: in this case, they indeed represent a discontinuity of the surface slope, that is, an angular point, and a different approach to coarsening should be used. [S1063-651X(98)07307-3]

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## I. INTRODUCTION

A crystalline surface growing under a flux of incoming particles from the vapor phase represents a typical example of an out-of-equilibrium system. Its microscopic evolution may be described as follows: once the adatom has arrived on the surface, it performs a thermally activated diffusion process until it is ''trapped'' somewhere, or it evaporates by coming back to the vapor phase. Which surface relaxation mechanism (surface diffusion or evaporation-condensation) indeed prevails depends on the temperature and the specific parameters of the material [1]. Anyway, for a wide class of materials (mainly metals), at the relevant temperatures for molecular beam epitaxy (MBE) desorption may be neglected. In this case, two different ''traps'' may be effective: another adatom (giving rise to a nucleation phenomenon), or a step. If we limit ourselves to the case of a high-symmetry surface, there are no preexisting steps and therefore the mentioned step belongs to a growing island. Once islands have coalesced, leading to the completition of one layer, steps should disappear and the whole previous process should start again.

The previous qualitative picture applies to the case of a stable layer-by-layer growth. In reality, it is hindered both by noise and by possible instabilities; sources of noise are fluctuations in the flux of incoming particles (shot noise), in the surface diffusion current (diffusion noise), and in the nucleation events (nucleation noise). While the first two have been well studied in the context of several different models [2], the latter one still needs a more basic comprehension [3]. Concerning deterministic instabilities, the main one responsible—and perhaps the sole one responsible for a homoepitaxial high-symmetry surface—for the destabilization of the flat surface is now known as the Ehrlich-Schwoebel effect [4]: an adatom approaching a step from above or below may have different probabilities of attachement. If the sticking from above is discouraged, an adatom that has the possibility to choose between two different kinds of steps (an ascending one and a descending one) will stick preferably to the ascending one, thus determining an uphill current. It is important to remark that this is a purely out-of-equilibrium effect, because at equilibrium detailed balance forbids such a current.

Even without entering into details, as will be done in the next section, it is possible to explain here the effect of such a mechanism. In fact, as first pointed out by Villain [5], the resulting surface current $j=\nu m$ ($m=\partial z/\partial x$ being the local slope of the surface and $z$ the local height), once put in the evolution equation $\partial z/\partial t=-\partial j/\partial x$ gives rise to a diffusion-type equation: $\partial_t z=-\nu\partial_x^2 z$, where the negative sign of the diffusion constant ($-\nu$) is responsible for the instability of the flat surface ($z=\text{const}$). In the present paper, we will mainly be concerned with the late stages of this instability, when additional and nonlinear terms must be introduced to describe the dynamics of the surface. In the next section, we will introduce a more general expression for the surface current $j$ and we will take into account the breaking of the $z\rightarrow -z$ symmetry, induced by the flux $F$ of atoms.

## II. THE SURFACE CURRENT

The study of a growth process may ideally be divided into two main steps: the first one starts from some microscopic

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point of view and should arrive to a continuum description of the surface/interface; the second one may even *assume* a given evolution equation and study it. Most of the difficulties encountered in a theoretical study of MBE are related to the first step. In the present paper, we will limit ourselves to a one-dimensional high-symmetry surface, and in this section we will introduce and justify a specific Langevin-type equation.

The local surface height $z(x,t)$ is generally supposed to satisfy a *local* equation of the form $\partial_t z = aF + \mathcal{F}(\partial_x z, \partial_x^2 z, \dots)$, $F$ being the incoming flux, $a$ the in-plane lattice constant, and $\mathcal{F}$ a function of the local profile of the surface (the out-of-plane lattice constant is set equal to one, i.e., $z$ is adimensional). The underlying hypotheses have been discussed in Ref. [6], where we have shown that the appearance of angular points in the surface profile may be treated correctly solely through the introduction of a nonlocal equation. We will take up this point again at the end of the article.

The flux $F$ contains a constant part $F_0$, which is ''eliminated'' by redefining $z(x,t)$: $z \to z - aF_0 t$, and a fluctuating part $\delta F(x,t)$ which represents the so-called shot noise, which is supposed to follow a Gaussian-like distribution:

$$\langle \delta F(x,t) \rangle = 0,$$

$$\langle \delta F(x,t)\, \delta F(x',t') \rangle = 2F_0 \delta(x-x')\, \delta(t-t'). \qquad (1)$$

In the limit of negligible desorption, and if overhangs are forbidden, surface growth proceeds by conserving both mass and volume: therefore, the function $\mathcal{F}$ must be derivable from a surface current $j$, and the evolution equation will be written in the form

$$\partial_t z(x,t) = -a\, \partial_x j + a\, \delta F(x,t). \qquad (2)$$

The central question is which current $j$ governs the evolution of the surface, a still debated question even for the simplified model of a one-dimensional surface, as shown by the following discussion on the different terms appearing in $j$. Symmetry arguments simply tell that $j$ does not depend on $z$ [7], but on its derivatives ($m = \partial_x z$, $m' = \partial_x^2 z$, $m'' = \partial_x^3 z, \dots$) and that—on a high-symmetry surface— it must be an odd function of $x$: so, a term proportional to $m$ or $m''$ satisfies this request, but if proportional to $m'$, it does not.

### A. Ehrlich-Schwoebel current

In the Introduction, we mentioned the Ehrlich-Schwoebel effect, which gives rise to a slope-dependent current: $j_{\mathrm{ES}}(m)$. Since it must be an odd function of $m$, its form at small $m$ will be $j_{\mathrm{ES}} = \nu m$. The coefficient $\nu$ depends [8,9] on the flux $F_0$, the diffusion length $l_D$, and the Schwoebel length $l_S$: $l_D$ measures the typical linear distance traveled by the ad- atom before meeting another one and forming the nucleus of a growing island. It represents the ''maximal'' size of a terrace, because if $l > l_D$ the probability to nucleate a new island on it is very high; during the first stages of growth, when the surface is still more or less flat, $l_D$ is also the typical size of a terrace. It is not so when an instability develops: in this case, $l$ may be much smaller than $l_D$ and the slope $m = 1/l$ may be fairly large. Indeed, the slope $1/l_D$ discriminates between a nucleation-dominated regime ($m \ll 1/l_D$) and a step-flow regime ($m \gg 1/l_D$): the latter is generally relevant for vicinal surfaces, which grow through sticking of adatoms to preexisting steps; anyway, if a flat surface develops an instability with regions of high slope, such a regime becomes important also for high-symmetry orientations.

The second relevant length, the Schwoebel length, is a measure of the asymmetry in the sticking coefficients of an adatom to a step. Its simplest form [1] is $l_S = a(D_-/D_+ - 1)$, where $D_+$ and $D_-$ are such coefficients for an adatom approaching the step from above ($D_+$) and below ($D_-$). The existence of an Ehrlich-Schwoebel effect means that $D_+ < D_-$ and therefore $l_S > 0$. To describe the meaning of $l_S$, let us consider a terrace of size $l$ ($< l_D$): if $l_S \ll l$, only a fraction $l_S/l$ of the fallen adatoms will contribute to the uphill current, and therefore $j_{\mathrm{ES}} = (F_0 l)(l_S/l) = F_0 l_S$, since the number of atoms arriving per unit time on a terrace of size $l$ is nothing but $F_0 l$. Conversely, if $l_S \gg l$ all the adatoms will stick to the ascending step, and so $j_{\mathrm{ES}} = F_0 l$. The simplest interpolation formula, valid for any value of $l_S$, is

$$j_{\mathrm{ES}} = \frac{F_0 l_S}{1 + l_S |m|}, \quad |m| > 1/l_D. \qquad (3)$$

This formula also allows one to obtain a semiquantitative expression for the parameter $\nu$: in fact, when $|m| \simeq 1/l_D$, Eq. (3) must match the expression valid at small slopes: $j_{\mathrm{ES}} = \nu m$. The result is $\nu = F_0 l_S l_D^2/(l_S + l_D)$. It is important to remark that all the previous considerations may be made more rigorous [6], but in this section we are mainly interested in justifying the expression for the current $j$, rather than in deriving it.

The main characteristic of the Ehrlich-Schwoebel current just discussed is that it has no zeros other than $m = 0$ and $m = \pm\infty$. A zero in $j_{\mathrm{ES}}$ is extremely important [10] because the other terms in $j$ will be seen to depend on higher-order derivatives of $z(x,t)$. So, a constant slope $m_0$ may be a stationary slope if and only if $j_{\mathrm{ES}}(m_0) = 0$. An extra zero $m_0$ may have different origins: the symmetry of the crystal lattice [10,11], nonthermal relaxation mechanisms [12], or a transient mobility of the adatom just after the deposition [13]. For example, the slope at 45° corresponds in a cubic lattice to the high-symmetry orientation (11): we expect that $j_{\mathrm{ES}}$ vanishes on it, as it vanishes on the (10) ($m = 0$) and (01) ($m = \infty$) orientations. A different example is the following: If atoms falling in the vicinity of a step have a higher probability to land on the lower terrace, or to kick down the step adatom, a downhill current $j_\downarrow$, proportional to the density of steps and therefore to the slope $m$, will appear: $j_\downarrow = -\nu' m$. So, if $\nu > \nu'$, a zero will appear when $(j_{\mathrm{ES}} + j_\downarrow) = 0$.

Whatever is the origin of extra zero(s) in the slope-dependent current, we can introduce two different models, according to the presence (model I) or absence (model II) of zeros at finite slope. The simplest expressions of $j_{\mathrm{ES}}$ for the two models, having the correct symmetry properties are [14]

$$\text{model I:} \quad j_{\mathrm{ES}} = \nu m (1 - m^2/m_0^2), \qquad (4)$$

$$\text{model II:} \quad j_{\rm ES}=\frac{\nu m}{1+l_D^2 m^2}. \tag{5}$$

Model II does not correspond to a phase separation process (see Sec. III): it will be discussed in Sec. VIII.

### B. Mullins-like current

The most famous ''equilibrium'' current is perhaps the one ($j_M$) introduced by Mullins [15] forty years ago, to study the relaxation towards equilibrium of a nonsingular grooved surface. A simple derivation starts from writing $j_M$ as the gradient of a chemical (surface) potential: $j_M=-\Gamma\partial_x\mu$, where $\Gamma$ is the adatom mobility, and afterwards to derive $\mu$ from a surface free energy:

$$\mu=\frac{\delta\mathcal{E}}{\delta z(x)}, \quad \text{with} \quad \mathcal{E}=\sigma\int dx\sqrt{1+m^2a^2}. \tag{6}$$

By combining the different equations, in the limit of small slopes we obtain

$$j_M=Km''(x) \tag{7}$$

with $K=a^2\Gamma\sigma$.

The usage of this expression in our problem may be questionable in at least two respects: First, it applies to a nonsingular surface, i.e., above the roughening transition $T_R$; second, it applies to a close-to-equilibrium surface. Concerning the first remark, our surface is a high-symmetry one and therefore almost necessarily below $T_R$, because for a high symmetry orientation the roughening temperature is equal to or nearly equal to the melting temperature $T_M$, while ordinary temperatures for MBE are well below $T_M$. Nevertheless, our surface—which is strongly out of equilibrium—contains a lot of steps because the incoming flux makes the surface rough [16]: therefore, the surface current should be nonsingular at zero slope.

The latter remark is more ''critical'': the Mullins current derives from thermal detachment of atoms from steps in order to minimize the surface free energy. It is not clear if such a process is effective in the presence of a flux $F$. For example, Stroscio and Pierce [17] state that thermal detachment is negligible in the homoepitaxial growth of Fe (at least at room temperature) and therefore they do not write [18] such a term in the current. Anyway, it has been shown [6,16] that the current (7) may derive also from nonequilibrium effects: nucleation noise and diffusion noise. The first one should be dominant and correspond to the value [6,16] $K=F_0 l_D^4$.

### C. Symmetry-breaking current

The terms in the surface current that have been introduced so far not only satisfy the $x\to -x$ symmetry [because $j(-x)=-j(x)$], but they also fulfill the up-down symmetry, corresponding to the change of sign of $z$. In fact, if $z\to -z$ both $j_{\rm ES}$ and $j_M$ change sign. However, there is no reason to expect that surface growth proceeds by conserving such symmetry, since the flux breaks it.

A symmetry-breaking (SB) term is intrinsically nonlinear, because any current of the form $j\sim\partial_x^n z(x,t)$ changes sign with $z$. The lowest order expression that changes sign with $x$ but does not change sign with $z$ is

$$j_{\rm SB}=\partial_x A(m^2), \tag{8}$$

where $A$ is any even function of the local slope. The simplest form for $A$, $A=(\lambda/2)m^2$, has been introduced by Sun *et al.* [19]. It is also called the ''conserved Kardar-Parisi-Zhang term,'' because in Eq. (2) it looks like the Laplacian of $(\partial_x z)^2$, i.e., the nonlinear term of the KPZ equation [20].

The current (8) is not derivable from a free energy. As pointed out by Somfai and Sander [21] it is necessary to rise the order of $j_{\rm SB}$ to make it derivable from some free energy (for example, $j_{\rm SB}\sim\partial_x[(m')^2]=(\delta/\delta m)\int dx\mathcal{F}_{\rm SB}$ with $\mathcal{F}_{\rm SB}\sim(m')^3$).

Before proceeding, let us discuss the physical origin of $j_{\rm SB}$. When there is a gradient in the density $\rho$ of adatoms, a current of the form $j=-D\partial_x\rho$ is expected, where $D$ is the diffusion constant. In the case of a growing surface, the applicability of the previous expression is not obvious, because steps are sinks for diffusing atoms and—at least if thermal detachment is forbidden—interlayer diffusion is absent. In spite of this, the above expression may help in understanding: in fact, adatom density on a terrace depends on its size $l$, because a larger terrace collects more atoms from the flux than a smaller one. So, $\rho=\rho(l)=\rho(|m|)$. In other words, the function $A$ appearing in $j_{\rm SB}$ seems to be proportional to the adatom density itself.

This interpretation can be made more rigorous for large slopes ($|m|=1/l>1/l_D$), where nucleation of new terraces is absent and $\rho$ can be simply determined by solving the diffusion equation $\partial_t\rho=F_0+D\partial_x^2\rho$ in the quasistatic approximation ($\partial_t\rho=0$) and with $\rho(0)=\rho(l)=0$ as boundary conditions (i.e., steps are perfect sinks). The resulting average density on the terrace is $\rho\simeq(F_0/D)l^2$ and the current is $j_{\rm SB}\simeq -F_0\partial_x(1/m^2)$. This expression agrees with those determined, with different methods, by Politi and Villain [6] and by Krug [22]. Hunt *et al.* [23] suggest that $j_{\rm SB}$ may derive from the sticking asymmetry induced by the Ehrlich-Schwoebel effect: nevertheless, $j_{\rm SB}$ does not vanish even if $l_S=0$ [6,22].

One could ask why the average value of $\rho$ is taken. The answer is that inhomogeneities in the adatom density on a given terrace give rise to $j_{\rm ES}$. In fact, if no Ehrlich-Schwoebel effect is present, $\rho(x)$ is symmetric with respect to the center of the terrace and therefore the average value of $\partial_x\rho$ vanishes. Conversely, if $l_S>0$ then $\langle\partial_x\rho\rangle_{\rm terrace}\neq 0$ and it corresponds just to $j_{\rm ES}$. This remark stresses the ''similar'' origin of $j_{\rm ES}$ and $j_{\rm SB}$. It is likely that a systematic derivation of the surface current should give all the terms we have introduced: $j_{\rm ES}$ (which depends on the slope $m$), $j_{\rm SB}$ (which depends on the curvature $m'$), and $j_M$ (which depends on a higher order derivative $m''$). Anyway, a rigorous derivation is still lacking at the moment, above all for a high-symmetry orientation.

### D. The current of our model

In the following, we will study the dynamical evolution of the surface, as determined by the current

$$j=j_{\rm ES}+j_M+j_{\rm SB}, \tag{9}$$

where

$$j_{\rm ES}=\nu m\left(1-\frac{m^2}{m_0^2}\right), \tag{10}$$

$$j_M=Km'', \tag{11}$$

$$j_{\rm SB}=\lambda mm'. \tag{12}$$

The reason for our choice is clear: We want to study the effect of the symmetry-breaking current ($j_{\rm SB}$) on the phase separation process determined by the other two terms of the surface current ($j_{\rm ES}+j_M$), and for this aim we choose the simplest expression for $j_{\rm ES}$—which must have a zero at a finite slope $m_0-$ and for $j_{\rm SB}$—for which we take $A(m^2)=\lambda m^2/2$. In the last section, we will discuss how the conclusions depend or *not* depend on the present choice.

## III. EVOLUTION IN THE ABSENCE OF THE SYMMETRY-BREAKING CURRENT

In the ''language'' of surface growth, the evolution of the surface proceeds as follows: After a time $t^*$ an instability of the flat surface with a well-determined wavelength $L^*$ develops. In this linear regime, $L^*$ is constant and the amplitude increases exponentially. Afterwards, because of the nonlinearity of $j_{\rm ES}$ a coarsening process takes place: the wavelength $L(t)$ of the moundlike (or pyramidlike) surface profile increases in time, while the maximal slope tends to the constant values $\pm m_0$. So, the surface is ''made up'' of neighboring regions where the slope is alternately (nearly) equal to $+m_0$ and $-m_0$.

The first stages of growth can be analyzed by linearizing Eq. (2) with the current (9):

$$a^{-1}\partial_t z(x,t)=-\nu\partial_x^2 z(x,t)-K\partial_x^4 z(x,t), \tag{13}$$

which shows [6,9,23] that the flat surface is unstable against deformations of wavelength larger than $L_c=2\pi\sqrt{K/\nu}$. The most unstable mode corresponds to $L_u=\sqrt{2}L_c$ and its amplitude grows as $\exp[(a\nu^2/4K)t]$. So, $L^*=L_u$ and $t^*=(4K/a\nu^2)$.

The nonlinear profiles of the mounds are determined as stationary solutions of Eq. (2), that is to say as solutions of the equation $j=0$:

$$j_{\rm ES}(m)+Km''(x)=0. \tag{14}$$

This equation can be derived by the following Lagrangian:

$$\mathcal{L}=(K/2)m'^2-V(m), \quad \text{with } V'(m)=j_{\rm ES}(m), \tag{15}$$

which corresponds to an anharmonic pendulum, once we have identified the slope $m$ as its spatial coordinate and $x$ as the time. Since the potential $V(m)=(\nu/2)m^2[1-m^2/2m_0^2]$ has two symmetric maxima in $\pm m_0$, the period of the oscillation (i.e., the wavelength of the surface profile) diverges when its amplitude (i.e., the maximal slope of the surface profile) goes to $m_0$. If $j_{\rm ES}$ followed model II, $V(m)$ would have no maxima and no limitation on the slope would be present.

By going on with this mechanical analogy, the existence of coarsening requires a condition on the stationary configurations: the period of the oscillation must be an increasing function of the amplitude [24]; a condition that is surely fulfilled by the potential $V(m)$, since the quartic correction has a negative sign. Clearly, coarsening also requires that these stationary solutions are not stable: more precisely, they must be unstable with respect to wavelength fluctuations, but stable with respect to amplitude fluctuations.

The previous mechanical analogy helps in understanding why the surface keeps a regular profile and also allows one to determine this profile at a given time, but it is not effective in determining the time dependence of $L(t)$, i.e., the coarsening law [23]. To this end, we must observe that the evolution equation for the local slope $m$ (which represents the ''order parameter'' of our problem) satisfies the noisy Cahn-Hilliard equation [25]:

$$a^{-1}\partial_t m=\partial_x^2\left(\frac{\delta\mathcal{F}}{\delta m}\right)+\eta(x,t), \quad \text{where } \mathcal{F}=\int dx\mathcal{L}. \tag{16}$$

This equation corresponds to a phase separation process, where the order parameter is conserved [$\partial_t\int dx m(x,t)=0$]. The system is made up of domains where $m$ equals one of the two degenerate minima of the potential energy $U(m)=-V(m)$; domains that are separated by domain walls move in order to minimize the ''action'' $\mathcal{F}$. Domain wall (or ''kink'') movement is determined both by their (deterministic) interaction and by fluctuations induced by the conserved noise. We will see that the growing surface (even in the presence of the symmetry-breaking current $j_{\rm SB}$) can be mapped in a one-dimensional system of interacting kinks that annihilate, so that the average distance $L(t)$ between kinks increases in time.

By using this method for the symmetric case ($j_{\rm SB}=0$), Kawasaki and Ohta [26] have found the equation of motion for the kinks, which has been then studied by Kawakatsu and Munakata [27]. The final result is that $L(t)$ grows logarithmically with time if noise is absent and grows as $t^{1/3}$ if noise is present.

## IV. KINK PROFILES

A stationary kink $M(x)$ is defined as a monotonic solution of $j[M(x)]=0$, with $M(x)$ tending to (different) minima of $U(m)$, when $x\to\pm\infty$. In the present case, there are only two symmetric minima in $\pm m_0$ and therefore only two kinks $M_\pm(x)$ are possible, the subscript corresponding to the sign of its first derivative, i.e., to the curvature of the surface profile.

The surprising result is that the ''shape'' of the kink does *not* change because of the introduction of the symmetry-breaking term. To see it, let us replace the expression

$$M_\pm(x)=\pm m_0\tanh(\kappa_\pm x/2) \tag{17}$$

in the differential equation $j=0$:

$$Km''(x)+\nu m(1-m^2/m_0^2)+\lambda mm'=0. \tag{18}$$

We obtain the following second degree equation for the parameters $\kappa_\pm$:

$$K\kappa_\pm^2 \mp \lambda m_0 \kappa_\pm - 2\nu = 0, \tag{19}$$

which gives the positive solutions

$$\kappa_\pm = (\sqrt{\lambda^2 m_0^2 + 8\nu K} \pm \lambda m_0)/2K. \tag{20}$$

Two limiting cases, corresponding to weak and strong symmetry breaking, will be frequently used:

$$\lambda m_0 \ll \sqrt{8\nu K}, \quad \kappa_+ = \kappa_- = \sqrt{2\nu/K} \equiv \kappa_0, \tag{21}$$

$$\lambda m_0 \gg \sqrt{8\nu K}, \quad \kappa_+ = \lambda m_0/K, \quad \kappa_- = 2\nu/\lambda m_0. \tag{22}$$

So, the effect of $j_{SB}$ is to create two classes of kinks: kinks ''$A$,'' given by the profile $M_+(x)$ and characterized by a width $(1/\kappa_+)$, and kinks ''$B$,'' given by the profile $M_-(x)$ and whose width is $(1/\kappa_-)$. For a strong $j_{SB}$, $\kappa_+ \gg \kappa_-$: kinks $A$ are much narrower than kinks $B$. It must also be observed that the product $(\kappa_+ \kappa_-)$ does not depend on $\lambda$, since it equals (see the algebraic equation) $(2\nu/K)$. In other terms, the effect of $j_{SB}$ is to shrink kinks $A$ and to widen kinks $B$, in such a way that the product of their widths keeps constant.

## V. FROM SURFACE DYNAMICS TO KINK DYNAMICS

In this section we will describe the method to solve the growth equation for the surface-slope profile:

$$a^{-1}\partial_t m = D_x[Km'' - U'(m) + \lambda m m'] \quad \text{with} \quad D_x = -\partial_x^2 \tag{23}$$

in a ''multikink'' approximation. Since our approach follows that introduced by Kawasaki and Ohta [26] to study the above equation in the absence of the $\lambda$ term, we will expose the main calculations in Appendix A and here we will limit ourselves to explaining the general lines of the method.

Once a kink is inserted in our problem, it moves with a given (constant) velocity $v^0$ and a profile $m(x,t) = M(x - v^0 t)$, where $v^0$ is found by solving the eigenvalues problem obtained by putting $m(x,t)$ in Eq. (23). Our system is made up of an ensemble of kinks $A$ that alternate to kinks $B$, and we will look for an approximate solution of Eq. (23) as a superposition of kinks centered in $x_i$ and moving with velocity $v_i$. Because of the interaction between kinks, $v_i$ is not a constant, and depends on the position of the other kinks. In principle, the nonlinear part of $U(m)$ (i.e., the quartic term $m^4$) gives rise to terms of $n$-kinks interaction: we will adopt a ''binary-interaction'' approximation, which will be further simplified by limiting to nearest-neighbor interaction. This procedure is justified by the fact that we are interested in the late stages of growth, when the distance between kinks is much larger than the width of their cores $(=1/\kappa_\pm)$: so, they interact only through the tails of the profiles, which means that the interaction decays exponentially, since $\tanh(\kappa x/2) \simeq \pm 1 \mp \exp(-\kappa|x|)$ when $x \to \pm\infty$. For the same reason, the velocities $v_i$ and the accelerations $\dot{v}_i$ will be considered ''small,'' because the typical size of the mounds grows slower than linearly: This means that the velocity of the coarsening process goes to zero, as time increases.

As a final result, we obtain a Langevin equation for the discrete variables $x_i(t)$, or—equivalently—for the kink-kink distances $X_i(t) \equiv x_{i+1}(t) - x_i(t)$, which will be studied by translating it in a Fokker-Planck equation.

The treatment of Eq. (23) (see Appendix A) gives the following coupled equations for the kink positions:

$$-2a^{-1}m_0^2 \sum_j (-1)^{i-j} |x_i - x_j| \dot{x}_j = (C_1) + (C_2) + (C_4) + \eta_i(t), \tag{24}$$

where

$$(C_1) = 8\nu m_0^2 [R_\beta(X_i) - R_\beta(X_{i-1})],$$
$$(C_2) = \beta(4/3) m_0^3 \kappa_\beta \lambda [R_{-\beta}(X_i) - R_{-\beta}(X_{i-1})], \tag{25}$$
$$(C_4) = -\beta 4 m_0^3 \kappa_\beta \lambda [R_\beta(X_i) - R_\beta(X_{i-1})],$$

and

$$\langle \eta_i(t) \rangle = 0,$$
$$\langle \eta_i(t)\, \eta_j(t') \rangle = -4m_0^2 F_0 (-1)^{i-j} |x_i - x_j| \, \delta(t - t').$$

Let us explain the notations: The $i$th kink is centered in $x_i$, and—because of the breaking of symmetry— two different classes of kinks exist. In accordance with Sec. IV, their profiles are given by $M_\beta(x) = \beta m_0 \tanh(\kappa_\beta x/2)$, where $\beta = \pm 1$. We will assume that the $i$th kink is of class $\beta$ (whatever is its value) and its nearest neighbors of class $-\beta$. The quantity

$$R_\beta(x) = \exp(-\kappa_\beta x) \tag{26}$$

in $(C_i)$ expresses the interaction between kinks, when the distances $|x_{i\pm1} - x_i|$ are large compared to $(1/\kappa_\beta)$.

Equation (24) can also be written in matrix form: $\boldsymbol{A}_{ij}\dot{x}_j = I_i + \eta_i$. The matrix $\boldsymbol{A}$ takes into account the kinematical coupling between kinks, due to the conservation of the order parameter, and $\underline{I}$ contains the forces between kinks. The matrix $\boldsymbol{A}$ can be inverted [27], giving a tridiagonal and symmetric $\boldsymbol{A}^{-1}$:

$$\boldsymbol{A}^{-1}_{ii} = \frac{a}{4m_0^2}\left(\frac{1}{X_i} + \frac{1}{X_{i-1}}\right), \tag{27}$$

$$\boldsymbol{A}^{-1}_{i+1,i} = \frac{a}{4m_0^2}\frac{1}{X_i}. \tag{28}$$

The evaluation of $\boldsymbol{A}^{-1}\underline{I}$ is trivial:

$$(\boldsymbol{A}^{-1}\underline{I})_i = \frac{a}{4m_0^2}\left(\frac{I_i + I_{i+1}}{X_i} + \frac{I_i + I_{i-1}}{X_{i-1}}\right) \tag{29}$$

and the explicit expression of $I_i$ is found directly from Eqs. (25):

$$I_i = R^*_\beta(X_i) - R^*_\beta(X_{i-1}), \tag{30}$$

where $R^*_\beta(X)$ is a linear combination of the two different $R_\beta(X)$:

$$R^*_\beta(X) \equiv c_\beta R_\beta(X) + d_\beta R_{-\beta}(X)$$

with

$$c_\beta = 8\nu m_0^2 - \beta 4 m_0^3 \kappa_\beta \lambda \quad \text{and} \quad d_\beta = \beta(4/3) m_0^3 \kappa_\beta \lambda .$$

Concerning the noise, it is preferable to work with quantities that are not spatially correlated. To this end, the matrix $\boldsymbol{A}^{-1}$ is written as the product $\boldsymbol{P}\boldsymbol{P}^T$ and new noise variables $\underline{\tilde{\eta}} = \boldsymbol{P}^T \underline{\eta}$ are defined. Since $\boldsymbol{P}$ is a bidiagonal matrix whose nonvanishing elements are

$$\boldsymbol{P}_{ii} = \boldsymbol{P}_{i+1,i} = \sqrt{\frac{a}{2m_0}} \frac{1}{\sqrt{X_i}}, \tag{31}$$

$\tilde{\eta}_i$ is given by $\tilde{\eta}_i = \sqrt{a}(\eta_i + \eta_{i+1})/(2m_0\sqrt{X_i})$, and it results that

$$\langle \tilde{\eta}_i(t) \rangle = 0, \quad \langle \tilde{\eta}_i(t) \tilde{\eta}_j(t') \rangle = 2aF_0 \delta_{ij} \delta(t-t'). \tag{32}$$

In order to eliminate the constant factor in the correlator, we simply put $\tilde{\eta}_i = \sqrt{2aF_0}\xi_i$. This way, the final equation for kink dynamics is

$$\dot{x}_i(t) = \frac{a}{4m_0^2}\left[\frac{I_i + I_{i+1}}{X_i} + \frac{I_i + I_{i-1}}{X_{i-1}}\right] + \frac{\sqrt{2F_0}a}{2m_0}\left[\frac{\xi_i}{\sqrt{X_i}} + \frac{\xi_{i-1}}{\sqrt{X_{i-1}}}\right], \tag{33}$$

$$\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t-t').$$

## VI. FORCES AND KINK VELOCITIES

In this section we want to discuss the effect of the symmetry breaking on the equations of motion for the kinks. Since we are here interested in the deterministic part of the interaction, we will not consider the noise. Therefore Eq. (24) takes the form $\boldsymbol{A}_{ij}\dot{x}_j = I_i$. Kawasaki and Ohta [26] suggest looking on $I_i$ as the force acting on the $i$th kink. Let us consider the two opposite limits: $\lambda = 0$ and $\lambda m_0 \gg \sqrt{8\nu K}$. For $R^*_\beta(X)$ we obtain

$$R^*_\beta(x) = 8\nu m_0^2 \exp(-\kappa_0 x) \tag{34}$$

in the first limit ($\lambda = 0$), and

$$R^*_+(x) = (4\lambda^2 m_0^4/3K)\exp(-\kappa_\lambda x),$$

$$R^*_-(x) = 16\nu m_0^2 \exp(-\kappa_\lambda x)$$

in the second one ($\lambda m_0 \gg \sqrt{8\nu K}$). In the previous equations, $\kappa_0 = \sqrt{2\nu/K}$ and $\kappa_\lambda = (\lambda m_0/K) \gg \kappa_0$. They correspond to $\kappa_-$ in the two pertinent limits.

In the case of the absence of the $\lambda$ term, we simply get

$$I_i = 8\nu m_0^2 [\exp(-\kappa_0 X_i) - \exp(-\kappa_0 X_{i-1})]. \tag{35}$$

This equation can be interpreted by saying that there is an attraction between kinks, proportional to $\exp(-\kappa_0 X)$. If $\lambda \neq 0$ (and ''strong''), then we must distinguish between positive and negative kinks:

$$I_i = \begin{cases} (4\lambda^2 m_0^4/3K)[\exp(-\kappa_\lambda X_i) - \exp(-\kappa_\lambda X_{i-1})], & \beta > 0 \\ 16\nu m_0^2 [\exp(-\kappa_\lambda X_i) - \exp(-\kappa_\lambda X_{i-1})], & \beta < 0. \end{cases} \tag{36}$$

The first comment is that symmetry breaking implies that a positive kink is attracted (by a negative one) more strongly than a negative kink is attracted by a positive one. In other words, if we assign a mass to a kink, a negative kink weighs more than a positive one, and the mass is proportional to the width of the kink itself.

This interpretation seems to be satisfactory, but if we analyze the velocities $\dot{x}_i(t)$ rather than the ''forces'' $I_i$ the picture becomes more complicated. In the limit $\lambda = 0$ we have

$$\dot{x}_i(t) = 2a\nu\left[\frac{\exp(-\kappa_0 X_i)}{X_{i-1}} - \frac{\exp(-\kappa_0 X_{i-1})}{X_i} + \frac{\exp(-\kappa_0 X_{i+1})}{X_i} - \frac{\exp(-\kappa_0 X_{i-2})}{X_{i-1}}\right]. \tag{37}$$

So, the effect of the conservation law (i.d. of the matrix $\boldsymbol{A}$) is that $\dot{x}_i(t)$ depends not only on the positions of the nearest-neighbor (nn) kinks ($x_{i\pm1}$), but also on those of the next-nearest (nnn) ones ($x_{i\pm2}$). Even more important, the nnn ''interaction'' is of the same order of magnitude as the nn one. While the interpretation of $\exp(-\kappa_0 X_i)$ and $\exp(-\kappa_0 X_{i-1})$ in Eq. (35), as, respectively, the interaction with the kinks $(i+1)$ and $(i-1)$, is straightforward, in Eq. (37) the generic term $\exp(-\kappa_0 X_l)$ is divided by a different $X_j$, and therefore a similar interpretation becomes less evident. Anyway, if we do not ascribe too much importance to the quantities $X_j$ in the denominator, Eq. (37) says that kink $i$ is attracted both by nn kinks and nnn kinks: the ''interaction'' between $i$ and $i \pm 2$ has a kinematical origin (conservation of the order parameter) and indeed depends on $x_{i\pm2} - x_{i\pm1}$ rather than on $x_{i\pm2} - x_i$. A further comment is that in the evaluation of $I_i + I_{i+1}$ two terms cancel exactly, because in this case action and reaction are opposite and equal.

If now we consider the case of a strong symmetry-breaking term, the velocity takes the form

$$\dot{x}_i(t)\big|_{\beta>0} = \frac{a\lambda^2 m_0^2}{3K}\left[\left(\frac{1}{X_i} + \frac{1}{X_{i-1}}\right)\exp(-\kappa_\lambda X_i) - \left(\frac{1}{X_i} + \frac{1}{X_{i-1}}\right)\exp(-\kappa_\lambda X_{i-1})\right] + 4a\nu\left[\frac{\exp(-\kappa_\lambda X_{i+1})}{X_i} - \frac{\exp(-\kappa_\lambda X_{i-2})}{X_{i-1}}\right] \tag{38}$$

and

$$\dot{x}_i(t)\big|_{\beta<0} = -\frac{a\lambda^2 m_0^2}{3K}\left[\frac{\exp(-\kappa_\lambda X_i)}{X_i} - \frac{\exp(-\kappa_\lambda X_{i-1})}{X_{i-1}}\right.$$

$$+\frac{\exp(-\kappa_\lambda X_{i+1})}{X_i}-\frac{\exp(-\kappa_\lambda X_{i-2})}{X_{i-1}}\Bigg]. \quad (39)$$

The surprising result is that the sign of the terms proportional to $\exp(-\kappa_\lambda X_i)$ and $\exp(-\kappa_\lambda X_{i-1})$ is inverted: so —because of kinematics—a negative kink is subject to a repulsive interaction with its nn kinks. This result derives from the unbalancing of action and reaction. A closer inspection of the derivation of Eqs. (38) and (39) allows to give the following interpretation: if $f_{ij}$ means the force exerted by the kink $j$ on the kink $i$ (so that $I_i=f_{i,i+1}+f_{i,i-1}$), then kinematics determines that the effective force $\tilde{f}_{i,i\pm1}$ is a linear combination of $(f_{i,i\pm1}+f_{i\pm1,i})$ and $f_{i,i\pm1}$. If $\lambda=0$, the first term vanishes, but if $\lambda\neq0$ it does *not*: Furthermore, for a negative kink $f_{i\pm1,i}$ prevails over $f_{i,i\pm1}$ and it corresponds to a repulsive force for kink $i$.

The conclusion we draw from the previous considerations is that negative kinks move much slower than positive kinks. This results on one side from the fact that a bigger mass can be attributed to them, and on the other side that they are subject to an effective repulsive nn interaction.

## VII. FROM KINK DYNAMICS TO COARSENING LAWS

The interesting dynamical variables are the kink-kink distances $X_i$, rather than the kink positions $x_i$. So, from Eq. (33) we obtain

$$\begin{aligned}\dot{X}_i(t)=\frac{a}{4m_0^2}\Bigg[&\frac{1}{X_{i+1}}[R^*_\beta(x_{i+2})+R^*_{-\beta}(x_{i+1})\\&-R^*_\beta(x_{i+1})-R^*_{-\beta}(x_i)]\\&-\frac{1}{X_{i-1}}[R^*_\beta(x_i)+R^*_{-\beta}(x_{i-1})\\&-R^*_\beta(x_{i-1})-R^*_{-\beta}(x_{i-2})]\Bigg]\\&+\frac{\sqrt{2F_0}a}{2m_0}\left[\frac{\xi_{i+1}}{\sqrt{X_{i+1}}}-\frac{\xi_{i-1}}{\sqrt{X_{i-1}}}\right]. \quad (40)\end{aligned}$$

The previous equations have the form

$$\dot{q}_i(t)=\mathcal{U}_i(\{q\})+\sum_j \mathcal{G}_{ij}(\{q\})\xi_j$$

$$\text{with}\quad \langle\xi_j(t)\xi_{j'}(t')\rangle=\delta_{jj'}\delta(t-t') \quad (41)$$

and we can therefore obtain a Fokker-Planck equation for the probability $\rho(\{q\},t)$ of finding a given distribution $\{q\}$, at time $t$. Two different procedures exist [28], due to Ito and to Stratonovich, but as remarked by Kawakatsu and Munakata [27] the result is the same. This is true even in the presence of the symmetry-breaking term, because the two procedures may differ with respect of the term $\mathcal{G}_{ij}$, which does not change if the $\lambda$ term is added.

The Fokker-Planck equation writes

$$\frac{\partial\rho}{\partial t}=-\sum_k\frac{\partial}{\partial q_k}[\mathcal{U}_k(\{q\})\rho]+\frac{1}{2}\sum_{kl}\frac{\partial^2}{\partial q_k\partial q_l}\sum_m \mathcal{G}_{km}\mathcal{G}_{lm}\rho. \quad (42)$$

Its actual form, in our case, is [29]

$$\begin{aligned}\frac{\partial\rho}{\partial t}=&-\sum_k\frac{\partial}{\partial X_k}[\mathcal{U}_k(\{X\})\rho]\\&+\frac{F_0a^2}{4m_0^2}\sum_k\frac{1}{X_k}\left[\frac{\partial^2}{\partial X_{k-1}^2}+\frac{\partial^2}{\partial X_{k+1}^2}-2\frac{\partial^2}{\partial X_{k-1}\partial X_{k+1}}\right]\rho, \quad (43)\end{aligned}$$

where $\mathcal{U}_k$ is nothing but the ''deterministic'' velocity of the $k$th kink.

We are interested in the time dependence of the average value of $X_i$ (which does not depend on $i$). To this end, we define the distribution functions:

$$g(X_i;t)=\int_0^\infty (dX)^{\check{}}_i\rho, \quad (44)$$

$$g_2(X_i,X_{i+1};t)=\int_0^\infty (dX)^{\check{}}_{i,\check{i+1}}\rho, \quad (45)$$

$$g_3(X_i,X_{i+1},X_{i+2};t)=\int_0^\infty (dX)^{\check{}}_{i,\check{i+1},\check{i+2}}\rho. \quad (46)$$

The notation $(dX)^{\check{}}_{i,\check{i+1},\ldots}$ means that the integration is performed on all the variables $X_j$ but $X_i, X_{i+1}, \ldots$ .

The details of the calculation follow Ref. [27] and therefore they will not be given here. By using the factorization approximation

$$g_2(X_i,X_{i+1};t)=g(X_i)g(X_{i+1}), \quad (47)$$

$$g_3(X_i,X_{i+1},X_{i+2};t)=g(X_i)g(X_{i+1})g(X_{i+2}) \quad (48)$$

and integrating Eq. (43) over $(dX)^{\check{}}_i$, we obtain

$$\frac{\partial g}{\partial t}=-\frac{\partial}{\partial X}J(X,t) \quad (49)$$

with the current of probability given by

$$\begin{aligned}J(X,t)=&\frac{a}{4m_0^2}\left\langle\frac{1}{X}\right\rangle[\langle R^*_+(X)+R^*_-(X)\rangle-(R^*_+(X)+R^*_-(X))]g\\&-\frac{F_0a^2}{2m_0^2}\left\langle\frac{1}{X}\right\rangle\frac{\partial g}{\partial X}. \quad (50)\end{aligned}$$

In the two relevant limits, $R^*_+(X)+R^*_-(X)$ takes the form

$$R^*_+(X)+R^*_-(X)$$

$$=\begin{cases}16\nu m_0^2\exp(-\kappa_0X), & \lambda=0 \quad (51)\\ (4\lambda^2m_0^4/3K)\exp(-\kappa_\lambda X), & \lambda m_0\gg\sqrt{\nu K} \quad (52)\end{cases}$$

and in the limit $\lambda=0$ we recover Eq. (4·4) of Ref. [27].

Important works on the solution of Eq. (49), which also go beyond the factorization approximation by taking into account correlations of consecutive domains, are given in a series of papers by Nagai and Kawasaki [30]. Here, we will follow Ref. [27] and the first of the papers cited in Ref. [30].

The time dependence of the density of kinks $n(t)$—or alternatively of the average kink-kink distance: $\overline{X}(t) \equiv \langle X \rangle = 1/n(t)$—is studied by assuming that at large times $\overline{X}$ represents the only relevant scale in the problem, and therefore $g(X;t)$ satisfies the scaling expression

$$g(X;t) = n(t)\tilde{g}(X/\overline{X}). \tag{53}$$

For example, for a Dirac-delta distribution (all the domains have the same size) $\tilde{g}(s) = \delta(s-1)$, and for a Poisson distribution (randomly distributed kinks) $\tilde{g}(s) = e^{-s}$.

Secondly, we will use a steady-state approximation [27] according to which the distribution $g(X;t)$ does not depend on time, on scales sufficiently small with respect to $\overline{X}(t)$: more precisely, on scales $X < X^*$. This means that the motion of a couple of kinks at a distance smaller than $X^*$ is essentially independent on the position of all the other kinks. Because of the scaling hypothesis, it must result that $X^* = \overline{X}/\alpha$, with $\alpha$ constant.

The temporal variation of $n(t)$ is determined by the number of kink-kink annihilations per unit time and unit length. Since each annihilation makes two kinks disappear, we have

$$\dot{n}(t) = 2n(t)J(X=0;t) = 2n(t)J(X^*;t), \tag{54}$$

where the second relation derives from the fact that $\partial_t g = 0$ implies $\partial_X J = 0$.

By approximating $\langle f(X) \rangle$ with $f(\overline{X})$ ($f$ is a generic function), and by neglecting $R^*_\beta(\overline{X})$ with respect to $R^*_\beta(X^*)$, we finally obtain the following expression for the current in $\overline{X}^*$:

$$J(X^*;t) = -\frac{a}{4m_0^2}\frac{1}{\overline{X}}[R^*_+(X^*) + R^*_-(X^*)]g(X^*) - \frac{F_0 a^2}{2m_0^2}\frac{1}{\overline{X}}\frac{\partial g}{\partial X}\bigg|_{X^*}. \tag{55}$$

#### 1. Deterministic regime

If the noise term is negligible,

$$J(X^*;t) = -\frac{a}{4m_0^2}\frac{1}{\overline{X}}[R^*_+(X^*) + R^*_-(X^*)]g(X^*). \tag{56}$$

Let us consider separately the two limiting cases. When $\lambda = 0$, by using Eq. (51), the current is written as

$$J(X^*;t) = -4a\nu\tilde{g}(1/\alpha)n^2(t)\exp(-\kappa_0/\alpha n) \tag{57}$$

and Eq. (54) becomes

$$\dot{n}(t) = -8a\nu\tilde{g}(1/\alpha)n^3(t)\exp(-\kappa_0/\alpha n), \tag{58}$$

whose solution gives, at large times

$$\overline{X}(t) \simeq (\alpha/\kappa_0)\ln(t/t_1), \quad t_1 = \left[\frac{e\alpha^2}{8\tilde{g}(1/\alpha)}\right]\frac{K}{a\nu^2} \quad (\lambda = 0). \tag{59}$$

In the opposite limit of a strong symmetry breaking ($\lambda m_0 \gg \sqrt{\nu K}$), a similar calculation gives

$$\overline{X}(t) \simeq (\alpha/\kappa_\lambda)\ln(t/t_2), \quad t_2 = \left[\frac{3e\alpha^2}{4\tilde{g}(1/\alpha)}\right]\frac{K}{a\nu^2} \quad (\lambda m_0 \gg \sqrt{\nu K}). \tag{60}$$

We therefore obtain that $t_1 \simeq t_2 \simeq t^*$, where $t^*$ was defined in Sec. III as the time necessary for the developing of the linear instability of the flat surface. So, the time scale for the logarithmic coarsening does not depend on $\lambda$, but the length scale does, since it depends on the width of the (largest) domain wall.

We can ask what is the meaning of the $\alpha$ dependence in Eqs. (59) and (60). As pointed out by Nagai and Kawasaki [30], since $\alpha \ln t = \ln t^\alpha$ the parameter $\alpha$ should have some ''universal'' value. In a mean-field calculation these authors find $\alpha = 1$, while in a numerical solution of the kink equations they obtain $\alpha \simeq 3.5$. More rigorous calculations [30] give $\alpha = 2.27$ if domains are completely uncorrelated, and $\alpha = 3.56$ if correlation effects between neighboring domains are taken into account.

#### 2. Noise-dominated regime

Now the current is

$$J(X^*;t) = -\frac{F_0 a^2}{2m_0^2}\frac{1}{\overline{X}}\frac{\partial g}{\partial X}\bigg|_{X^*}. \tag{61}$$

The equation for $n(t)$ is written as

$$\dot{n}(t) = -\left[\frac{F_0 a^2}{m_0^2}\tilde{g}'(1/\alpha)\right]n^4(t) \tag{62}$$

and the solution is

$$\overline{X}(t) = \overline{X}_0(t/t_0)^{1/3}, \quad \overline{X}_0 = \left[\frac{3a\tilde{g}'(1/\alpha)}{m_0^2}\right]^{1/3}, \quad t_0 = 1/F_0 a. \tag{63}$$

So, we will have logarithmic coarsening at ''small'' times and a powerlike one at later times. The crossover time is determined by the relation $(\alpha/\kappa)\ln(t_c/t^*) = \overline{X}_0(t_c/t_0)^{1/3}$. By neglecting the logarithmic dependence (also because $t^* \gg t_0$), it is found that

$$t_c \approx t_0\left(\frac{\alpha}{\kappa\overline{X}_0}\right)^3. \tag{64}$$

So, the ratio between the crossover time in the presence of a strong asymmetry and the crossover time in the absence of the $\lambda$ term is approximately given by

$$\frac{t_c(\lambda m_0 \gg \sqrt{\nu K})}{t_c(\lambda=0)} \approx \left(\frac{\kappa_0}{\kappa_\lambda}\right)^3 = \left(\frac{\lambda m_0}{\sqrt{\nu K}}\right)^3. \tag{65}$$

It is important to stress the cubic exponent in the previous expression: even a not large value of $(\kappa_0/\kappa_\lambda)$ gives rise to a logarithmic coarsening that proceeds for a much longer time, because kink interaction is stronger and therefore a larger $t_c$ is necessary so that noise gets the better of the deterministic regime.

We want to emphasize that in the noise-dominated regime, the actual value of $\alpha$ is much less relevant than in the deterministic regime, because of the power-law character of the coarsening.

## VIII. DISCUSSION

The main result of the present paper is that ''coarsening laws'' do not change if the symmetry-breaking current $j_{SB}$ is put in the problem (at least, as far as a continuum local description is valid: see below). This is mainly due to the fact that the functional form of the kinks does not change, as shown by the exact solution we have given in Sec. IV for their profile.

So, a first question is how general is this result if we modify the surface current, and therefore Eq. (18). A first obvious modification would be to replace $\partial_x A(m^2)=\lambda m m'$ with a more complicated expression of the slope $m$. This corresponds to having a $\lambda$ depending on $m$; in fact, $\lambda=\lambda(m^2)=2A'(m^2)$. Since in the late stages of growth the slope is almost everywhere equal to $\pm m_0$, $\lambda$ is almost everywhere a constant equal to $\lambda(m_0^2)$. Is it possible to simply replace $\lambda$ by $\lambda(m_0^2)$ in the final results? This should not be a bad approximation, as suggested by the analysis of Eq. (18) when $\lambda$ depends on $m$. In fact, the asymptotic behavior of $M(x)$ (the relevant one for kink interaction) and the values of $\kappa_\pm$ can be found by linearizing the differential equation with respect to $m_0-M(x)$ for a positive kink and to $-m_0+M(x)$ for a negative kink (in both cases, in the limit $x\to\infty$). Because of the linearization, only the value $\lambda(m_0^2)$ enters in the problem and therefore determines the profile.

In a similar way, we can take into account a possible $m$ dependence of the quantity $K$. In this case, such a dependence might arise from a slope-dependent mobility $\Gamma$ [31]—if $K$ has an equilibrium origin—or from the dependence on the terrace length $l$ of the probability to nucleate a new terrace [6], if $K$ derives from nucleation noise.

Let us now discuss the choice of the slope-dependent current: $j_{ES}=\nu m(1-m^2/m_0^2)$. The only features we require to have a phase separation process are $j'_{ES}(m=0)>0$ (to make the flat surface unstable) and $j_{ES}(m_0)=0$ for some finite value $m_0$ [indeed, $m_0$ must be the first zero of $j_{ES}$]. These features define the so-called model I.

Modifications of $j_{ES}$ inside this model do not change the given picture, as suggested by the analysis of the stationary profile of the kink (for the sake of simplicity we put $\lambda=0$). If we linearize the equation

$$j_{ES}(m)+Km''(x)=0 \tag{66}$$

with respect to $\epsilon(x)=m_0-m(x)$, we obtain

$$j'_{ES}(m_0)\epsilon(x)+K\epsilon''(x)=0, \tag{67}$$

whose solution is again an exponential function. So, for $x\to\infty$, $m(x)=m_0-\epsilon_0 e^{-\kappa x}$, with $\kappa=\sqrt{-j'_{ES}(m_0)/K}$. In our expression of $j_{ES}$ [Eq. (9)], $j'_{ES}(m_0)=-2\nu$ and $\kappa$ reduces to $\kappa_0=\sqrt{2\nu/K}$.

Conversely, in model II there is no finite zero in $j_{ES}$. This implies that the slope increases with no upper limit: for $\lambda=0$, as shown by Hunt *et al.* [23], the maximal slope $M_0$ in the profile is asymptotically proportional to the size of the mounds: $M_0(t)\sim\bar{X}(t)$. Since the potential energy $U(m)$ $[U'(m)=-j_{ES}(m)]$ has no minima, it is no more possible to define domains and domain walls, i.e., kinks.

Concerning the time dependence of coarsening, the only existing numerical results are the ones found by Hunt *et al.* [23]. According to their simulations (in the presence of noise), $\bar{X}(t)\approx t^n$ with $n\simeq 0.22$, a fairly small value [32]. No (rigorous) theoretical derivation of $n$ is available at the moment. Some scaling arguments—applicable to noiseless growth—can be found in Rost and Krug [33] and in Golubović [34]: The former give $n\leqslant 1/4$ while the latter gives the equality $n=1/4$ [35].

A final question we want to face now is how narrow kinks $A$ actually are. In the limit $\lambda m_0 \gg \sqrt{\nu K}$, from Eq. (22) we have $\kappa_+=\lambda m_0/K$ and $\kappa_-=2\nu/\lambda m_0$. A simple inspection shows that $[\lambda]=[K]=\text{length}^3\ \text{time}^{-1}$. Previous evaluations suggest [6,16] $\lambda\approx K\approx F_0 l_D^4$. This expression for $K$ is surely wrong if thermal detachment plays an important role. Conversely, if $\lambda$ and $K$—or, more precisely, $\lambda(m_0^2)$ and $K(m_0^2)$—are of the same order of magnitude, we obtain $\kappa_+\approx m_0$. This means that the width of the positive kink $(=1/\kappa_+)$ is nothing but the inverse of the value of the constant slope in the surface profile: so, if $m_0$ is determined by the symmetry of the crystal lattice, $m_0\simeq 1/a$ and the positive kink is as narrow as a lattice constant. In this case, our description would break down, because the regions of positive curvature in the surface profile would correspond to a discontinuity of the slope, i.e., to angular points, that are not compatible with a local continuum equation [36].

## IX. CONCLUSIONS

The kink picture not only has allowed one to find the coarsening law in the presence of the symmetry-breaking term, but it has also given a qualitative description of the dynamics that allows a better comprehension of the evolution of the system: the widening or the narrowing of a kink, the consequent different velocities of kinks $A$ and $B$, the conservation of the order parameter seen as a kinematical constraint on kink movement, the difference between the ''real'' force acting on a kink and the ''effective'' force felt by the kink, because of such constraint.

In this respect, the most important consequence of the breaking of symmetry is that negative kinks feel an effective *repulsive* interaction with the nn kinks (but attractive with the nnn ones). It is important to stress this point because coarsening is the result of a global attraction between kinks: if kinks repelled each other, the configuration with $X_i=\text{const}$ would be stable.

Finally, the kink picture has provided the condition of

applicability of the local theory:

$$\frac{1}{\kappa_+}=\frac{K(m_0^2)}{m_0\lambda(m_0^2)}\gg a. \tag{68}$$

If this relation is not fulfilled, a different method to study coarsening should be used. In Ref. [6] we showed that in this case the evolution of the surface is governed by a nonlocal current; alternatively, we can keep a local description, but we must add a singular term to the current $j$, and couple the Langevin equation $\partial_t z(x,t)=-a\partial_x j$ with specific evolution equations for the angular points. It would be clearly interesting to check if a different coarsening process may arise from an ''angular point'' picture.

To our knowledge, the current (9) has not been formerly studied. The closest model is the one considered by Stroscio *et al.* [18] in two dimensions, where the Mullins term $[Km''(x)]$ is replaced by a higher order one $[Km''''(x)]$ and the resulting equation is studied numerically. Clearly, in two dimensions analytical treatments are much more difficult; anyway, a numerical solution of the model studied in the present paper is available at the moment. One reason is that in two dimensions, even the model without $j_{\rm SB}$ is not yet fully understood, since the evolution equation for $m(x,t)$ [37] is no longer equivalent to the Cahn-Hilliard equation.

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## APPENDIX A: LANGEVIN EQUATIONS FOR THE KINKS

### 1. Absence of noise

The starting point is the following multikink expansion,

$$m(x,t)=M_i(x,t)+\sum_{j>i}[M_j(x,t)-M_j(-\infty)]+\sum_{j<i}[M_j(x,t)-M_j(\infty)] \tag{A1}$$

$$\equiv M_i(x,t)+\delta m_i \tag{A2}$$

which gives rise, once replaced in Eq. (23), to

$$a^{-1}\sum_j\left[-v_jM_j'+\dot v_j\frac{\partial M_j}{\partial v_j}\right]=D_xj[m(x,t)]. \tag{A3}$$

$M_j$ depends on $x$ and $t$ through the combination $(x-v_jt)$ and $M_j'$ is the derivation with respect to all of this argument. We will also use the notation $d_xM$ to mean the same kind of derivation. The single kink profile is found by simply dropping the sum $\Sigma_j$ and the term in $\dot v$ in Eq. (A3):

$$-a^{-1}v_j^0M_j'=D_xj[M_j]. \tag{A4}$$

It will be useful to consider, together with $M_j$, also its spatial derivative $M_j'$ which is localized around $x=x_j$. We define also $\tilde M_j'$ through the relation $M_j'(x)=D_x\tilde M_j'(-x)$. They satisfy the relations

$$-a^{-1}v_j^0M_j'(x)=D_x[Kd_x^2-U''(M_j)+\lambda M_j'+\lambda M_jd_x]\times M_j'(x), \tag{A5}$$

$$a^{-1}v_j^0\tilde M_j'(x)=[Kd_x^2-U''(M_j)+\lambda M_j'+\lambda M_jd_x]D_x\tilde M_j'(x). \tag{A6}$$

Now, let us multiply (A3) by $\tilde M_i'(x)$ and integrate on $x$. By defining $\delta v_j\equiv v_j-v_j^0$, we can write:

$$a^{-1}\sum_j\int dx\tilde M_i'(x)\left\{-\delta v_jM_j'+\dot v_j\frac{\partial M_j}{\partial v_j}\right\}=\int dx\tilde M_i'D_xj[m]+a^{-1}\sum_j v_j^0\int dx\tilde M_i'M_j'. \tag{A7}$$

The next step is to replace $m(x,t)=M_i(x)+\delta m_i$ in the current $j[m]$. The definition of the nonlinear part of the potential $U(m)$ [or equivalently of the current $U'(m)$] is self-explanatory.

$$\begin{aligned}j[m]&=Km''-U'(m)+\lambda mm'\\&=KM_i''+K\delta m_i''-U'(M_i+\delta m_i)+\lambda(M_i+\delta m_i)(M_i'+\delta m_i')\\&=KM_i''+K\delta m_i''-U'(M_i)-U''(M_i)\delta m_i-U_{NL,i}'\\&\quad+\lambda M_iM_i'+\lambda\delta m_iM_i'+\lambda M_i\delta m_i'+\lambda\delta m_i\delta m_i'.\end{aligned} \tag{A8}$$

The three terms that do not depend on $\delta m_i$ once Eq. (A4) is used cancel the term $j=i$ in the last summation of Eq. (A7); $\delta m_i''$ is simply written as $\Sigma_{j\neq i}M_j''$, while all the other terms for the moment remain unchanged. So, the right hand side of Eq. (A7) is rewritten as

$$\mathrm{RHS}|_{(\mathrm{A7})}=\int dx\tilde M_i'(x)\left\{\sum_{j\neq i}[KD_xM_j''+a^{-1}v_j^0M_j']-D_x[U_{NL,i}'+U''(M_i)\delta m_i]+\lambda D_x[\delta m_iM_i'+M_i\delta m_i'+\delta m_i\delta m_i']\right\}. \tag{A9}$$

Let us consider separately some terms:

$$\int dx\tilde{M}_i'(x)\sum_{j\neq i} KD_xM_j''=\int dxK\delta m_i d_x^2 D_x\tilde{M}_i', \tag{A10}$$

$$-\int dx\tilde{M}_i'(x)D_x[U''(M_i)\delta m_i]=-\int dxU''(M_i)\delta m_i D_x\tilde{M}_i', \tag{A11}$$

$$\lambda\int dx\tilde{M}_i'D_x[\delta m_i M_i'+\delta m_i' M_i]=\lambda\int dx[\delta m_i M_i'+2M_i\delta m_i']D_x\tilde{M}_i'+\lambda\int dx\delta m_i[M_i'+M_i d_x]D_x\tilde{M}_i'. \tag{A12}$$

Equation (A9) is therefore rewritten as

$$\begin{aligned}\mathrm{RHS}|_{(A7)}=&\int dx\delta m_i[Kd_x^2-U''(M_i)+\lambda M_i'+\lambda M_i d_x]D_x\tilde{M}_i'+\int dx\tilde{M}_i'\sum_{j\neq i} a^{-1}v_j^0M_i'\\&+\int dx\tilde{M}_i'[-D_xU_{NL,i}'+\lambda D_x(\delta m_i M_i'+2M_i\delta m'+\delta m_i\delta m_i')]\equiv(A)+(B)+(C).\end{aligned} \tag{A13}$$

This way, Eq. (A7) takes the form (LHS≡left-hand side): $\mathrm{LHS}|_{(A7)}=(A)+(B)+(C)$.

By using Eq. (A6),

$$(A)=a^{-1}\int dx\tilde{M}_i'\sum_{j\neq i}[-v_i^0M_i'], \tag{A14}$$

which can be summed to $(B)$, giving

$$(A)+(B)=a^{-1}\int dx\tilde{M}_i'\sum_j (v_j^0-v_i^0)M_j' \tag{A15}$$

and subtracting $\mathrm{LHS}|_{(A7)}$,

$$(A)+(B)-\mathrm{LHS}|_{(A7)}=-(C), \tag{A16}$$

that is to say:

$$\begin{aligned}a^{-1}\sum_j\left[(v_j-v_i^0)(\tilde{M}_i',M_j')-\dot{v}_j\left(\tilde{M}_i',\frac{\partial M_j}{\partial v_j}\right)\right]=&\int dx\tilde{M}_i'[D_xU_{NL,i}'-\lambda D_x(\delta m_i M_i'+2M_i\delta m'+\delta m_i\delta m_i')]\\&\equiv(C_1)+(C_2)+(C_3)+(C_4).\end{aligned} \tag{A17}$$

In the previous equation we have used the following scalar product:

$$(R,S)=\int_{-\infty}^{+\infty}dxR(x)S(x). \tag{A18}$$

The three terms in square brackets on the right-hand side $[(C_2)+(C_3)+(C_4)]$ represent the effect of the symmetry-breaking current.

By integrating by parts and by using the definition of $\tilde{M}_i'$,

$$(C_1)=(M_i'(-x),U_{NL,i}'). \tag{A19}$$

If we define the function $G(x,y)\equiv U'(x+y)-U'(x)-yU''(x)$, then $U_{NL,i}'=G(M_i,\delta m_i)$. In the following, we will also make use of the function $\tilde{U}(x,y)\equiv U(x+y)-U(x)-yU'(x)$. It is obvious that $G(x,y)=\partial_x\tilde{U}(x,y)$.

We observe that (i) $G(M_i,0)=0$; (ii) $G$ may be written as a Taylor expansion whose generic term contains $(\delta m_i)^n$; (iii) $G$ is not linear in $\delta m_i$, but if we use the binary interaction approximation, it is indeed linear. This approximation corresponds to

$$(\delta m_i)^n\approx\sum_{j>i}[M_j-M_j(-\infty)]^n+\sum_{j<i}[M_j-M_j(\infty)]^n. \tag{A20}$$

In this approximation, we obtain

$$(C_1)=\sum_{j>i}\int dx M_i'(x_i-x)G(M_i(x-x_i),M_j-M_j(-\infty))+\sum_{j<i}\int dx M_i'(x_i-x)G(M_i(x-x_i),M_j-M_j(\infty)). \quad \text{(A21)}$$

We must observe that $M_i'$ is not vanishing only when $x \approx x_i$; furthermore, $[M_j-M_j(\pm\infty)]$ goes to zero when $(+)$ $x>x_j$ or $(-)$ $x<x_j$. On the basis of these considerations, it is possible to write

$$(C_1)=\sum_{j>i}\int_{x_j}^{+\infty} dx M_i'(x_i-x)G(M_i(x-x_i),\Delta M_j) + \sum_{j<i}\int_{-\infty}^{x_j} dx M_i'(x_i-x)G(M_i(x-x_i),-\Delta M_j), \quad \text{(A22)}$$

where $\Delta M_j \equiv M_j(\infty)-M_j(-\infty)$.

Since $M_i'(x)$ is an even function of $x$:

$$\int_{x_a}^{x_b} dx M_i'(x_i-x)G(M_i(x-x_i),\text{const}) = \tilde{U}(M_i(x-x_i),\text{const})\Big|_{x_a}^{x_b} \quad \text{(A23)}$$

and $(C_1)$ can be written as

$$(C_1)=\sum_{j>i}[\tilde{U}(M_i(\infty),\Delta M_j)-\tilde{U}(M_i(x_j-x_i),\Delta M_j)] + \sum_{j<i}[\tilde{U}(M_i(x_j-x_i),-\Delta M_j) - \tilde{U}(M_i(-\infty),-\Delta M_j)]. \quad \text{(A24)}$$

At the first order in the small quantities $[M_i(x_j-x_i)-M_i(\pm\infty)]$ ($\pm$ respectively for $j>i$ and $j<i$), we have

$$(C_1)=-\sum_{j>i}[M_i(x_j-x_i)-M_i(\infty)]G(M_i(\infty),\Delta M_j) + \sum_{j<i}[M_i(x_j-x_i)-M_i(-\infty)]G(M_i(-\infty),-\Delta M_j). \quad \text{(A25)}$$

In the following, we will restrict ourselves to nearest-neighbor kinks interaction, and therefore only the terms $j=i\pm1$ will survive in Eq. (A25). If we also use the fact that

$$G(M_i(\pm\infty),\pm\Delta M_{i\pm1})=\mp\Delta M_{i\pm1}U''(M_i(\pm\infty)) \quad \text{(A26)}$$

we obtain the following final expression:

$$(C_1)=[M_i(x_{i+1}-x_i)-M_i(\infty)]\Delta M_{i+1}U''(M_i(\infty)) + [M_i(x_{i-1}-x_i)-M_i(-\infty)]\Delta M_{i-1}U''(M_i(-\infty)). \quad \text{(A27)}$$

The procedure to follow for the treatment of the other terms $(C_i)$ is similar. In other words, if $R(x)$ and $S(x)$ are functions that are localized respectively in $x_1$ and $x_2$, we make the approximation

$$R(x-x_1)S(x-x_2)\approx R(x-x_1)S(x_1-x_2)+R(x_2-x_1)\times S(x-x_2) \quad \text{(A28)}$$

and then we retain only the term corresponding to the function decreasing more rapidly [for example, if $R(x)$ was a Dirac delta function, only the first term would be retained, because the second one would be exactly zero]. We give here only the results.

$$(C_2)=\lambda\{[M_{i+1}(x_{i+1}-x_i)-M_{i+1}(\infty)]-[M_{i-1}(x_i-x_{i-1})-M_{i-1}(\infty)]\}\int_{-\infty}^{+\infty} dx (M_i')^2,$$

$$(C_3)=0,$$

$$(C_4)=-\frac{\lambda}{2}(\Delta M_{i+1})^2 M_i'(x_{i+1}-x_i)+\frac{\lambda}{2}(\Delta M_{i-1})^2 \times M_i'(x_i-x_{i-1}).$$

The expression $(C_3)=0$ means that such a term is always of higher order than the others.

From now on, notation must take into account explicitly the existence of two different classes of kinks. By using the following results

$$M_\beta(x)-M_\beta(\infty)\simeq-\beta 2m_0\exp(-\kappa_\beta x) \quad \text{when} \quad x\to\infty$$

$$R_\beta(x)\equiv\exp(-\kappa_\beta x),$$

$$\Delta M_i=\beta 2m_0,$$

$$U''(\pm m_0)=2\nu,$$

$$\int_{-\infty}^{+\infty} dx[M_\beta'(x)]^2=\frac{2}{3}m_0^2\kappa_\beta$$

it is straightforward to write

$$(C_1)=8\nu m_0^2[R_\beta(X_i)-R_\beta(X_{i-1})],$$

$$(C_2)=\beta(4/3)m_0^3\kappa_\beta\lambda[R_{-\beta}(X_i)-R_{-\beta}(X_{i-1})], \quad \text{(A29)}$$

$$(C_4)=-\beta 4m_0^3\kappa_\beta\lambda[R_\beta(X_i)-R_\beta(X_{i-1})].$$

We can now put together Eq. (A29) with the previous ones. A further approximation is to neglect the ''deforma-

tion'' of the kink profile, due to its velocity, and to suppose that kinks are immobile in the absence of interactions. This way, we obtain

$$a^{-1}\sum_j v_j(\tilde{M}_i',M_j')=(C_1)+(C_2)+(C_4), \quad \text{(A30)}$$

where the LHS can be further developed:

$$\begin{aligned} a^{-1}\sum_j v_j(\tilde{M}_i',M_j') &= a^{-1}\sum_j v_j\int dx\tilde{M}_i'(x)M_j'(x) \\ &= a^{-1}\sum_j v_j\int dx D_x\tilde{M}_i'(x)D_x^{-1}M_j'(x) \\ &= a^{-1}\sum_j v_j\int dx M_i'(x)D_x^{-1}M_j'(x). \end{aligned} \quad \text{(A31)}$$

We therefore have to determine the inverse of the operator $D_x$. By following Kawasaki and Ohta [26]

$$D_x^{-1}A(x,t)=-\frac{1}{2}\int dx'|x-x'|A(x',t). \quad \text{(A32)}$$

The ''integration constants'' appearing when the operator $D_x$ is inverted are shown to be irrelevant for the kink dynamics (Ref. [27]).

By applying Eq. (A32) to Eq. (A31):

$$\int dxM_i'(x)D_x^{-1}M_j'(x)$$

$$=-\frac{1}{2}\int\int dxdx'M_i'(x)|x-x'|M_j'(x') \quad \text{(A33)}$$

$$\approx-\frac{1}{2}|x_i-x_j|\int\int dxdx'M_i'(x-x_i)M_j'(x-x_j) \quad \text{(A34)}$$

$$=-\frac{1}{2}|x_i-x_j|\Delta M_i\Delta M_j. \quad \text{(A35)}$$

So, Eq. (A17) is finally

$$-a^{-1}\frac{\Delta M_i}{2}\sum_j \Delta M_j|x_i-x_j|\dot{x}_j=(C_1)+(C_2)+(C_4). \quad \text{(A36)}$$

### 2. The effect of noise

The term of noise $\delta F(x,t)$ in Eq. (2) corresponds to a term $\eta(x,t)=\partial_x\delta F(x,t)$ on the right-hand side of Eq. (A3). To see how it affects the kink movement, it must be multiplied by $\tilde{M}_i'(x)$ and integrated on $x$. Since the LHS of Eq. (A36) indeed corresponds to minus the LHS of Eq. (A3), if we call $\eta_i(t)$ the noise term to be added to $(C_1)+(C_2)+(C_4)$ in Eq. (A36), it will result in

$$\eta_i(t)=-\int dx\tilde{M}_i'(x)\,\eta(x,t)=\int dx\,\partial_x\tilde{M}_i'(x)\,\delta F(x,t). \quad \text{(A37)}$$

The following properties are found [27]:

$$\langle\eta_i(t)\rangle=0 \quad \text{(A38)}$$

and

$$\begin{aligned} \langle\eta_i(t)\,\eta_j(t')\rangle &= 2F_0\delta(t-t')\int dx\tilde{M}'(x)D_x\tilde{M}_j'(x) \\ &= -4m_0^2F_0(-1)^{i-j}|x_i-x_j|\,\delta(t-t'). \end{aligned} \quad \text{(A39)}$$

To derive the spatial correlation between noise, we have used the definition of $\tilde{M}_i'$ and inverted the operator $D_x$. Finally, we have used the fact that $\Delta M_i\Delta M_j=4m_0^2(-1)^{i-j}$, a relation that can be used also for the LHS of Eq. (A36). So, we obtain the following system of coupled Langevin equations:

$$\begin{aligned} -2a^{-1}m_0^2\sum_j (-1)^{i-j}|x_i-x_j|\dot{x}_j \\ =(C_1)+(C_2)+(C_4)+\eta_i(t). \end{aligned} \quad \text{(A40)}$$


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Current (4) has been used in the context of surface growth by Stroscio *et al.* [18]. The fact that it diverges when $|m|\to\infty$ is not relevant, because $m=m_0$ is a ''stable fixed point.'' Current (5), which is correct only in the limit of a strong Ehrlich-Schwoebel effect [see Eq. (3)], has been introduced in $2+1$ dimensions by M. D. Johnson, C. Orme, A. W. Hunt, D. Graff, J. Sudijono, L. M. Sander, and B. G. Orr, Phys. Rev. Lett. **72**, 116 (1994).

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